Notes on the Vertical Normal Modes of the Atmosphere

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Contents

1 Introduction

The purpose of these notes is to give a detailed derivation of the *vertical normal modes* of the atmosphere. By “vertical”, we mean that we are interested in understanding the motions of a fluid (atmosphere) that has a non-constant vertical temperature and stratification profile. By “normal modes”, we mean that we seek to understand a complex system by identifying the essential elements of its motions. We do this by assuming some degree of linearity, in this case by considering a system that is perturbed around a motionless background state.

Throughout the notes, exercises are provided to test understanding. Exercises marked with a dot indicate results that are used later in the document. Readers are therefore encouraged to read over, if not complete, the exercises.

2 The Vertical Structure Operator

We begin with the 3D primitive equations for a statically stable atmosphere in log-pressure coordinates $\tilde{z}$ [1]:

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial \Phi}{\partial x}, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{\partial \Phi}{\partial y}, \tag{2.1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial \tilde{z}}\left(\rho_0 \tilde{w}\right) = 0, \tag{2.1c}$$

$$\frac{\partial \Phi}{\partial \tilde{z}} = \frac{RT}{H_s}, \tag{2.1d}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \tilde{z}}\right) + N^2 \tilde{w} = 0, \tag{2.1e}$$

where $\tilde{z}$ is our log-pressure vertical coordinate; $\tilde{w}$ is the vertical velocity in our log-pressure coordinates; $\rho_0(\tilde{z})$ is a fixed reference density profile; T is the temperature perturbation from a background profile $T_0(\tilde{z})$; $N^2(\tilde{z}) > 0$ is the buoyancy frequency, a measure of the background stratification; and Φ is the geopotential. The other variables take their standard meteorological meanings.

We also impose rigid lid boundary conditions at our a height $\tilde{z} = \tilde{z}_B$ and a top height $\tilde{z} = \tilde{z}_T$

$$\tilde{w}(x, y, t; \tilde{z}_B) = 0, \tag{2.2a}$$

$$\tilde{w}(x, y, t; \tilde{z}_T) = 0, \tag{2.2b}$$

Exercise 2.1 The first two equations of (2.1) are the zonal and meridional momentum equations. What are the corresponding names for the other three?

Our goal is to better understand the vertical behavior of this model. Therefore, it would be useful to collapse all the vertical structure information into a single term. The vertical coordinate appears in (2.1c)-(2.1e). These equations involve u, v, w, T, and Φ. The horizontal equations are just in terms of u, v, and Φ, so we should try to eliminate w and T by writing them in terms of the other three variables.

[1] In log-pressure coordinates, the vertical coordinate is expressed as $\tilde{z} = -H_s \ln(p/p_s)$, where p_s is the pressure that defines $\tilde{z} = 0$ and $H_s = R\overline{T_0}/g$ is a reference scale height of the atmosphere, set by the average value of the basic-state temperature profile $\overline{T_0}$.

From the thermodynamic equation (2.1e) we have

$$\tilde{w} = \frac{-1}{N^2}\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \tilde{z}}\right). \tag{2.3}$$

Multiplying by ρ_0 and taking the vertical derivative gives

$$\frac{\partial}{\partial \tilde{z}}\left(\rho_0 \tilde{w}\right) = -\frac{\partial}{\partial \tilde{z}}\left(\frac{\rho_0}{N^2}\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \tilde{z}}\right)\right). \tag{2.4}$$

By the continuity equation (2.1c), this is also equal to

$$\frac{\partial}{\partial \tilde{z}}(\rho_0 \tilde{w}) = -\rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{2.5}$$

Equating these two expressions gives

$$-\rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\partial}{\partial \tilde{z}}\left(\frac{\rho_0}{N^2}\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \tilde{z}}\right)\right). \tag{2.6}$$

Rearranging,

$$\frac{1}{\rho_0}\frac{\partial}{\partial \tilde{z}}\left(\frac{\rho_0}{N^2}\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial \tilde{z}}\right) - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{2.7}$$

Assuming that N^2 is constant in time, we can pull the time derivative out to the front of the vertical derivatives and re-arrange to get

$$\frac{\partial}{\partial t}\left[\frac{1}{\rho_0}\frac{\partial}{\partial \tilde{z}}\left(\frac{\rho_0}{N^2}\frac{\partial \Phi}{\partial \tilde{z}}\right)\right] - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{2.8}$$

Note that the bracketed term contains all the vertical derivatives now. We can thus define an operator [2] L, which we term the *vertical structure operator*, which acts on a function $\Phi(\tilde{z})$ as

$$\boxed{L[\Phi] = \frac{1}{\rho_0}\frac{\partial}{\partial \tilde{z}}\left(\frac{\rho_0}{N^2}\frac{\partial \Phi}{\partial \tilde{z}}\right).} \tag{2.9}$$

Thus, our system of equations can now be written as

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial \Phi}{\partial x}, \tag{2.10a}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{\partial \Phi}{\partial y}, \tag{2.10b}$$

$$\frac{\partial}{\partial t}L[\Phi] = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{2.10c}$$

[2] An operator is a map that takes a function as an input and returns another function as an output. A common example is the derivative operator d/dx, which maps a function $f(x)$ to its derivative $f'(x)$.

3 Vertical Normal Modes as an Eigenbasis

3.1 What good would an eigenfunction do?

In writing (2.10), our zonal and meridional momentum equations are the same as before, but we have now combined all the vertical derivatives into a single term, $L[\Phi]$. To understand how the vertical behavior of this system, then, we need to understand how L acts on functions.

Exercise 3.1 Show that L is a linear operator. That is, for any functions $f(\tilde{z})$ and $g(\tilde{z})$, and constants α and β show that

$$L[\alpha f(\tilde{z}) + \beta g(\tilde{z})] = \alpha L[f] + \beta L[g] \tag{3.1}$$

A good starting point to solving any problem is to think about what conditions would make the problem easier to solve. In that spirit, let's suppose (for now, just suppose) that there were, somewhere in the world, some function $\psi(\tilde{z})$ such that

$$L[\psi] = -\lambda\psi, \tag{3.2}$$

for some constant $\lambda \in \mathbb{R}$ (the reason for the negative sign in Equation 3.2 will become apparent soon). Such a function ψ would be called an "eigenfunction" of L, and λ its corresponding "eigenvalue" [3]

Exercise 3.2 Show that if $\psi(\tilde{z})$ is an eigenfunction of L with corresponding eigenvalue λ, then for any constant c, the scaled function

$$\varphi(\tilde{z}) = c\psi(\tilde{z})$$

is also an eigenfunction with eigenvalue λ.

While we are supposing things, let's also suppose that this function ψ were to also give us the vertical structures of u, v, and Φ. That is, suppose (again, just supposing for now) that

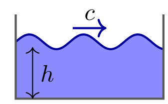
$$u(x, y, \tilde{z}, t) = \hat{u}(x, y, t)\psi(\tilde{z}) \tag{3.3a}$$

$$v(x, y, \tilde{z}, t) = \hat{v}(x, y, t)\psi(\tilde{z}), \tag{3.3b}$$

$$\Phi(x, y, \tilde{z}, t) = \hat{\Phi}(x, y, t)\psi(\tilde{z}). \tag{3.3c}$$

[3] The root word "eigen" mean "own", or "self" in German. The notion of an eigenfunction is analogous to notion of an eigenvector $\vec{v}$ of a matrix A, which satisfies $A\vec{v} = c\vec{v}$ for some constant c.

If all of this were so, then we could write our system of equations (2.10) as



$c = \sqrt{gh} = \lambda^{-1/2}$

$$\frac{\partial \hat{u}}{\partial t} - f\hat{v} = -\frac{\partial \hat{\Phi}}{\partial x}, \tag{3.4a}$$

$$\frac{\partial \hat{v}}{\partial t} + f\hat{u} = -\frac{\partial \hat{\Phi}}{\partial y}, \tag{3.4b}$$

$$\frac{\partial \hat{\Phi}}{\partial t} + \frac{1}{\lambda}\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y}\right) = 0. \tag{3.4c}$$

Exercise 3.3 Using (3.2), and assuming u, v, and Φ have the form given by (3.3), derive Equations (3.4a)-(3.4c).

We observe that the system (3.4) is of the same form as the shallow water equations for a layer of incompressible fluid. In this pseudo-shallow water system, the role of the gravity wave speed c is played by the $\lambda^{-1/2}$. Since in a shallow water system the gravity wave speed is given by $c = \sqrt{gh}$, where h is the depth of the fluid, we can define a parameter called the *equivalent depth* as

$$\boxed{h_e = \frac{1}{g\lambda}} \tag{3.5}$$

Let us summarize what we have shown up till now. We have shown that

- if $L[\Phi]$ had an eigenfunction $\psi(\tilde{z})$ with corresponding eigenvaue λ; **and**
- if the vertical structures of u, v, and Φ were all given by that eigenfunction ψ;
- **then** our original three-dimensional system would behave like a two-dimensional incompressible fluid, whose gravity wave speed is given by $\lambda^{-1/2}$.

The important point here is that, when the above conditions are satisfied, the whole systems behaves *as though it had no vertical dimension at all.* This is a massive simplification of the dynamics. So given that the existence of these eigenfunctions would make our lives much easier, we are motivated to ask

1. Do such eigenfunctions exist? How many are there?
2. When can the vertical structure of u and v be given by eigenfunctions of Φ?
3. How do the eigenvalues depend on the physical parameters of our problem?
4. How can we compute these eigenfunctions?

In the remainder of this section, we will answer Questions 1 and 2. We will answer Question 3 by explicitly computing the eigenfunctions in the case that N^2 is a constant. Then we will answer Question 4 by discretizing our problem into a generalized eigenvalue problem.

3.2 About these eigenfunctions

We now consider the question: how many eigenfunctions does L have, and what properties do they possess? This is a purely mathematical question that depends on L itself. The answer to this question is given by "Sturm-Liouville" theory, which is concerned with finding functions $y(x)$ and numbers λ that satisfy the differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = -\lambda w(x)y \tag{3.6}$$

defined on a bounded interval $[a, b]$, where $p(x) > 0$ and $w(x) > 0$ for all $x \in [a, b]$, and where the problem satisfies the boundary conditions

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0, & \alpha_1, \alpha_2 \text{ not both } 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0, & \beta_1, \beta_2 \text{ not both } 0. \end{cases} \tag{3.7}$$

Exercise 3.4 Show that (2.10c) can be written in the form of (3.6).

• **Exercise 3.5** Show that the rigid-lid boundary conditions (2.2a) imply that the eigenfunctions of L satisfy the following boundary conditions:

$$\left.\frac{d\psi}{d\tilde{z}}\right|_{\tilde{z}_B} = \left.\frac{d\psi}{d\tilde{z}}\right|_{\tilde{z}_T} = 0 \tag{3.8}$$

Verify that these boundary conditions are of the form (3.7).
Hint: Use the thermodynamic equation (2.1e) *and separation of variables* $\Phi(x, y, z, t) = \phi(x, y, t)\psi(z)$

Since our problem is of the form (3.6), we can import the following facts about our vertical structure operator L, which we will state without proof:

1. **L has infinitely many real eigenvalues and eigenfunctions**: There exists an infinite sequence of non-negative real numbers

$$0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \tag{3.9}$$

with $\lambda_n \to \infty$ as $n \to \infty$, and a corresponding set of functions $\{\psi_i(\tilde{z})\}_{i=0}^{\infty}$, that are distinct up to scaling by a constant, such that

$$L[\psi_n(\tilde{z})] = -\lambda_n \psi_n(\tilde{z}) \tag{3.10}$$

2. **The eigenfunctions of L are orthogonal under a weighted product**: If we write the mass weighted integral product of two functions $f(\tilde{z})$ and $g(\tilde{z})$ as

$$\langle f, g\rangle = \int_{\tilde{z}_B}^{\tilde{z}_T} \rho_0(\tilde{z}) f(\tilde{z}) g(\tilde{z})\ d\tilde{z} \tag{3.11}$$

Then the eigenfunctions $\{\psi_i\}$ satisfy

$$\langle \psi_m, \psi_n \rangle = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{3.12}$$

A consequence of this is that the eigenfunctions are *linearly independent*: a weighted sum of them vanishes only if every coefficient vanishes,

$$\sum_{i=0}^{N} c_i \psi_i = 0 \iff c_i = 0 \text{ for every } i. \tag{3.13}$$

3. **The eigenfunctions of L form a complete basis**. We can expand any[4] function $f(\tilde{z})$ in our *eigenbasis* as

$$f(\tilde{z}) = \sum_{i=0}^{\infty} \hat{f}_i \psi_i(\tilde{z}) \tag{3.14}$$

Where the real numbers $\{\hat{f}_i\}$ are given as

$$\hat{f}_i = \langle f(\tilde{z}), \psi_i(\tilde{z}) \rangle \tag{3.15}$$

These facts about L will allow us to describe out three-dimensional system (2.1) as a combination of two-dimensional shallow water systems of the form (3.4).

• **Exercise 3.6** Use Exercise 3.2 to argue that we can always scale eigenfunctions ψ_n to satisfy

$$\langle \psi_n, \psi_n \rangle = 1 \tag{3.16}$$

When a function satisfies (3.16), we say that the function has *unit length.*

• **Exercise 3.7** Show that the eigenvalues of L are non-negative.
Hint: Compute $\langle \psi, L[\psi] \rangle$ for some eigenfunction ψ via integration by parts. You will need to use the assumption that $N^2 > 0$, as well as the boundary conditions (3.8) *(Exercise 3.5).*

3.3 Expanding the 3D equations

Since the eigenbasis is complete, we can expand u, v, and Φ in this basis:

[4]There are pathological functions one can propose that would not satisfy this, but they are not likely to appear in any physically meaningful setting.

$$u(x,y,\tilde{z},t) = \sum_{n=0}^{\infty} \hat{u}_n(x,y,t)\psi_n(\tilde{z}) \tag{3.17a}$$

$$v(x,y,\tilde{z},t) = \sum_{n=0}^{\infty} \hat{v}_n(x,y,t)\psi_n(\tilde{z}) \tag{3.17b}$$

$$\Phi(x,y,\tilde{z},t) = \sum_{n=0}^{\infty} \hat{\Phi}_n(x,y,t)\psi_n(\tilde{z}) \tag{3.17c}$$

Where

$$\hat{u}_n(x,y,t) = \langle u, \psi_n \rangle \tag{3.18a}$$

$$\hat{v}_n(x,y,t) = \langle v, \psi_n \rangle \tag{3.18b}$$

$$\hat{\Phi}_n(x,y,t) = \langle \Phi, \psi_n \rangle \tag{3.18c}$$

Using (3.17), we can write (2.10a) as

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} \hat{u}_n\psi_n\right) - f\left(\sum_{n=0}^{\infty} \hat{v}_n\psi_n\right) = -\frac{\partial}{\partial x}\sum_{n=0}^{\infty} \Phi_n\psi_n \tag{3.19}$$

We can re-arrange this to give us

$$\sum_{n=0}^{\infty}\left(\frac{\partial \hat{u}_n}{\partial t} - f\hat{v}_n + \frac{\partial \hat{\Phi}_n}{\partial x}\right)\psi_n = 0 \tag{3.20}$$

A similar expansion of (2.10b) gives us

$$\sum_{n=0}^{\infty}\left(\frac{\partial \hat{v}_n}{\partial t} + f\hat{u}_n + \frac{\partial \hat{\Phi}_n}{\partial y}\right)\psi_n = 0 \tag{3.21}$$

Expanding (2.10c),

$$(\mathbf{??})\frac{\partial}{\partial t}L\left[\sum_{n=0}^{\infty} \hat{\Phi}_n\psi_n\right] = \frac{\partial}{\partial x}\sum_{n=0}^{\infty} \hat{u}_n\psi_n + \frac{\partial}{\partial y}\sum_{n=0}^{\infty} \hat{v}_n\psi_n \tag{3.22}$$

Using the facts that $L[\hat{\Phi}_n\psi_n] = \hat{\Phi}_n L[\psi_n]$ and $L[\psi_n] = -\lambda_n\psi_n$, we can simplify (**??**) to

$$\sum_{n=0}^{\infty}\left(-\lambda_n\frac{\partial \hat{\Phi}_n}{\partial t} - \frac{\partial \hat{u}_n}{\partial x} - \frac{\partial \hat{v}_n}{\partial y}\right)\psi_n = 0 \tag{3.23}$$

Using (3.13), we can conclude that each parenthetical quantity in (3.20), (3.21), and (3.23) is zero: for $n = 0, 1, 2, ...$, we then have

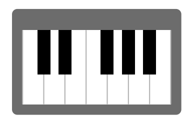
$$\frac{\partial \hat{u}_n}{\partial t} - f\hat{v}_n = -\frac{\partial \hat{\Phi}_n}{\partial x} \tag{3.24a}$$

$$\frac{\partial \hat{v}_n}{\partial t} + f\hat{u}_n = -\frac{\partial \hat{\Phi}_n}{\partial y} \tag{3.24b}$$

$$\frac{\partial \hat{\Phi}_n}{\partial t} + \frac{1}{\lambda_n}\left(\frac{\partial \hat{u}_n}{\partial x} + \frac{\partial \hat{v}_n}{\partial y}\right) = 0 \tag{3.24c}$$

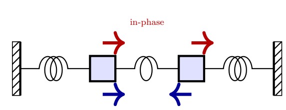
The system (3.24) is, like (3.4), formally similar to the shallow water equations. However, we have an infinite family of such systems, each corresponding to a given eigenfunction. Each vertical mode propagates horizontally at a phase speed $c_n = 1/\sqrt{\lambda_n} = \sqrt{gh_{e,n}}$, where $h_{e,n} = 1/g\lambda_n$ is the *equivalent depth* corresponding to the eigenfunction ψ_n.

3.4 Interpreting the vertical modes



What we have done is shown that our fully 3D systems (2.1) can be understood as a linear combination of 2D shallow water systems. Each one of these shallow water systems corresponds to a particular vertical structure, given by an eigenfunction of our vertical structure operator L.

A good analogy to what we have done is to think of a piano. A piano is capable of producing arbitrarily complicated music. However, no matter how complicated the music, we can always express the music as a combination of different piano keys being played in different combinations. In the same way, the arbitrarily complicated dynamics of our full 3D system (2.1) (the piano) can be expressed as the combinations of 2D systems (the piano keys). Furthermore, the piano keys are independent, in the sense that one cannot make the piano play, say, middle C by hitting some combination of other keys. This corresponds to the linear independence of our vertical modes (3.13). We will explore the extent to which this assumption, which is the property that makes our model a *linear* model, is justified when considering the real atmosphere in a later point in our notes.



It's worth noting that this type of analysis is by no means unique to atmospheric science. The idea of decomposing a complicated system into a sum of independent, simpler pieces arises across physics. The canonical example is a pair of masses coupled by springs (two masses connected to each other and to two walls by three springs). Its motion can always be written as a combination of two modes: a low-frequency mode in which both masses oscillate in phase, moving together, and a higher-frequency mode in which they oscillate out of phase, moving toward and away from each other. Each mode oscillates at its own frequency, independently of the other — which is precisely the finite-dimensional version of what we have found here, where each vertical mode ψ_n evolves as its own independent shallow-water system. The only difference is that the spring system has two degrees of freedom and hence two modes, whereas our operator L has infinitely many. This method is referred to as *normal mode analysis*, with the corresponding eigenfunctions being called *normal modes*. This motivates the terminology of calling the n^{th} eigenfunction ψ_n as the n^{th} *vertical normal mode*.

4 Computing the Eigenfunctions

In this section we will actually compute the normal modes (eigenfunctions of L). We will first discuss the barotropic mode, which can be computed by the time-honored method of "calculation by inspection". Next, we will consider the case where $N^2(\tilde{z})$ is a constant function. Finally, we will discuss how these normal modes can be computed numerically for a given (discrete) N^2 profile, as might be obtained from reanalysis or a similar gridded product.

4.1 The barotropic mode ψ_0

We are interested in finding functions $\psi(\tilde{z})$ such that

$$L[\psi(\tilde{z})] = \frac{1}{\rho_0}\frac{\partial}{\partial \tilde{z}}\left(\frac{\rho_0}{N^2}\frac{\partial \psi(\tilde{z})}{\partial \tilde{z}}\right) = -\lambda\psi(\tilde{z}) \tag{4.1}$$

for some number $\lambda \in \mathbb{R}$. I've made the dependence of ψ on $\tilde{z}$ explicit to emphasize that we seek *vertical functions* that can satisfy this equation. We know from Exercise 3.7 that any such $\lambda \geq 0$. Guided by this, we may ask: **is there any function that has an eigenvalue of $\lambda = 0$?**

In fact, there is! Inspection of our operator L shows that any constant function $\psi(\tilde{z}) = C$ would satisfy (4.1), with $\lambda = 0$. Since we can always scale our eigenfunctions to have unit length (Exercise 3.6), we take our eigenfunction corresponding to λ_0 to be

$$\boxed{\psi_0(\tilde{z}) = \frac{1}{\sqrt{\int_{\tilde{z}_B}^{\tilde{z}_T} \rho_0(\tilde{z})\, d\tilde{z}}} = \text{Constant for all } \tilde{z}} \tag{4.2}$$

. What can we say about this mode?

- **This mode is always present:** For any basic state profiles $\rho_0(\tilde{z})$ and $N^2(\tilde{z})$, ψ_0 is an eigenfunction with eigenvalue $\lambda_0 = 0$.
- **This mode is the lowest mode** ψ_0 From Exercise 3.7 we know that $\lambda = 0$ is the lowest value that an eigenvalue can take. Thus, $\lambda_0 = 0$ is the lowest eigenvalue of L, and ψ_0 can correspondingly be called the "gravest" normal mode of L.
- **This mode has no vertical structure:** Since ψ_0 is a constant for all heights, our corresponding horizontal wind field is given by

$$u_0(x, y, \tilde{z}, t) = \hat{u}_0(x, y, t)\psi_0(\tilde{z}) = \text{Constant} \times \hat{u}_0(x, y, t) \tag{4.3}$$

A similar argument for v then implies

$$\frac{\partial u}{\partial \tilde{z}} = \frac{\partial v}{\partial \tilde{z}} = 0 \iff \text{No vertical shear in horizontal winds} \tag{4.4}$$

Thus, our horizontal wind field that corresponds to this normal mode is constant with height.

- **This mode is barotropic:** The corresponding geopotential field for ψ_0 is given by

  $$\Phi_0(x, y, \tilde{z}, t) = \hat{\Phi}_0(x, y, t)\psi_0(\tilde{z}) = \text{Constant} \times \hat{\Phi}_0(x, y, t) \tag{4.5}$$

  By the hydrostatic relation (2.1d), this implies that there are no temperature perturbations, so the temperature depends only on $\tilde{z}$. Since $\tilde{z}$ is a monotonic function of pressure, the implies that the temperature also depends only on pressure. This is the definition of a barotropic state.

- **This mode has no vertical motion:** By the thermodynamic equation (2.1e),

  $$\tilde{w} \propto \frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \tilde{z}}\right) \tag{4.6}$$

  Since corresponding geopotential field for ψ_0 is given by

  $$\Phi_0 = \hat{\Phi}_0(x, y, t)\psi_0(\tilde{z}), \tag{4.7}$$

  the corresponding vertical velocity $\tilde{w}_0$ is given by

  $$\tilde{w}_0 \propto \frac{\partial \hat{\Phi}_0}{\partial t} \times \frac{\partial \psi_0}{\partial \tilde{z}} = 0 \tag{4.8}$$

  Thus the barotropic mode corresponds to motions that are purely horizontal and without any vertical component.

- **This mode corresponds to an infinite equivalent depth**: We defined the *equivalent depth* of a vertical mode as the depth of a shallow water system that posses equal wave propagation speed (3.5). When $\lambda = 0$, the corresponding equivalent depth, and thus wave speed, approaches infinity. A way to interpret this peculiar result is to say that any horizontal perturbations in the geopotential field are smoothed infinitely fast by this mode. In other words, this mode cannot support propagating gravity waves at all.

  An equivalent way of viewing this is to re-write (3.24c) as

  $$\lambda_n \frac{\partial \Phi_n}{\partial t} = -\left(\frac{\partial \hat{u}_n}{\partial x} + \frac{\partial \hat{v}_n}{\partial y}\right) \tag{4.9}$$

  For $\lambda_0 = 0$, we thus obtain that the flow is horizontally non-divergent. A layer of fluid that is infinitely deep cannot be stretched or squeezed by any finite amount, and is thus non-divergent; this gives us the $h_e \to \infty$ result.

4.2 The baroclinic modes for constant N^2

For modes that are higher order than the barotropic mode ($n > 0$), we have

$$\frac{\partial \Phi_n}{\partial \tilde{z}} = \hat{\Phi}_n(x, y, t) \frac{\partial \psi_n}{\partial \tilde{z}} \neq 0 \tag{4.10}$$

Thus, by the hydrostatic equation (2.1d), we have non-zero temperature perturbation from the basic state temperature profile $T_0(\tilde{z})$ along a given surface $\tilde{z} =$ const. Therefore, temperature depends both on pressure (through $\tilde{z}$) and on density (by the ideal gas law). This is the definition of a baroclinic state, and motivates us to call the vertical normal mode $\psi_{n \geq 1}$ as the n^{th} *baroclinic mode*.

In general, since L (and thus its eigenfunctions) depend on $N^2(\tilde{z})$, we cannot compute a closed form expression for the baroclinic modes. However, in the case that $N^2(\tilde{z})$ is a constant, then we can obtain analytical expressions for the baroclinic modes.

4.2.1 Deriving expressions for the baroclinic modes

For a constant N^2, our eigenproblem (4.1) becomes

$$L[\psi] = \frac{1}{\rho_0 N^2} \frac{\partial}{\partial \tilde{z}} \left(\rho_0 \frac{\partial \psi}{\partial \tilde{z}} \right) = -\lambda \psi \qquad (N^2 = \text{const}) \tag{4.11}$$

By the product rule, (4.11) becomes

$$\frac{1}{\rho_0 N^2} \left(\frac{\partial \rho_0}{\partial \tilde{z}} \frac{\partial \psi}{\partial \tilde{z}} + \rho_0 \frac{\partial^2 \psi}{\partial \tilde{z}} \right) = -\lambda \psi \tag{4.12}$$

In log-pressure coordinates, the density decays exponentially with $\tilde{z}$:

$$\rho_0(\tilde{z}) = \rho_s e^{-\tilde{z}/H_s} \tag{4.13}$$

Therefore, we have

$$\frac{1}{\rho_0 N^2} \left(\frac{-\rho_0}{H_s} \frac{\partial \psi}{\partial \tilde{z}} + \rho_0 \frac{\partial^2 \psi}{\partial \tilde{z}} \right) = -\lambda \psi \tag{4.14}$$

Rearranging, we obtain the second order (ordinary) differential equation

$$\frac{d^2 \psi}{d\tilde{z}^2} - \frac{1}{H_s} \frac{d\psi}{d\tilde{z}} + \lambda N^2 \psi = 0 \tag{4.15}$$

Where we have replaced the partial differential symbols ∂ with regular differentials to emphasize that ψ is solely a function of $\tilde{z}$. We can solve this equation by assuming a solution of the form

$$\psi(\tilde{z}) = e^{r\tilde{z}} \tag{4.16}$$

Substituting (4.16) into (4.15) gives us

$$\left(r^2 - \frac{1}{H_s}r + \lambda N^2\right) e^{r\tilde{z}} = 0 \tag{4.17}$$

Solving for values of r for which the parenthetical quantity in (4.17) vanishes gives two solutions:

$$r_\pm = \frac{1}{2H_s} \pm \sqrt{\frac{1}{4H_s^2} - \lambda N^2} \tag{4.18}$$

Thus, we have a general solution of the form

$$\psi(\tilde{z}) = A\exp(r_+\tilde{z}) + B\exp(r_-\tilde{z}) \tag{4.19}$$

- **Exercise 4.1** Show that when $0 < \lambda N^2 < \frac{1}{4H_s^2}$ (so $0 < r_- < r_+$), the only solution of the form (4.19) that satisfies the boundary conditions (3.8) is the zero function $\psi \equiv 0$. *Note: the same holds at the cutoff $\lambda N^2 = \frac{1}{4H_s^2}$, where the roots coincide and the solution becomes $\psi = (A + B\tilde{z})\exp(\tilde{z}/2H_s)$; the boundary conditions again force $A = B = 0$, so no mode lives there either.*

- **Exercise 4.2** Show that $\lambda = 0$ gives roots $r_- = 0$, $r_+ = \frac{1}{H_s}$, and that in this case the only solution of the form (4.19) that satisfies the boundary conditions (3.8) is the constant (barotropic) solution.

Non-constant (baroclinic) solutions can only exist in the regime $\lambda N^2 > \frac{1}{4H_s^2}$ (Exercises 4.1 and 4.2). In this regime, $r_\pm$ take on complex values. Writing

$$\frac{1}{4H_s^2} - \lambda N^2 = -\left(\lambda N^2 - \frac{1}{4H_s^2}\right) = -m^2 \tag{4.20}$$

where $m^2 > 0$, we can write (4.18) as

$$r_\pm = \frac{1}{2H_s} \pm im \tag{4.21}$$

where $i^2 = -1$, and

$$m = \sqrt{\lambda N^2 - \frac{1}{4H_s^2}} > 0. \tag{4.22}$$

The general solution (4.19) then becomes

$$\psi(\tilde{z}) = e^{\tilde{z}/2H_s}(Ae^{im\tilde{z}} + Be^{-im\tilde{z}}) \tag{4.23}$$

Where $A = (\alpha_R + i\alpha_I)$ and $B = (\beta_R + i\beta_I)$ are now complex numbers with real and imaginary components $\alpha_R, \beta_R \in \mathbb{R}$ and $\alpha_I, \beta_I \in \mathbb{R}$, respectively.

Substituting $A = \alpha_R + i\alpha_I$ and $B = \beta_R + i\beta_I$ and applying Euler's identity $e^{\pm im\tilde{z}} = \cos(m\tilde{z}) \pm i\sin(m\tilde{z})$,

$$\psi(\tilde{z}) = e^{\tilde{z}/2H_s}\Big[(\alpha_R+i\alpha_I)\big(\cos(m\tilde{z})+i\sin(m\tilde{z})\big)+(\beta_R+i\beta_I)\big(\cos(m\tilde{z})-i\sin(m\tilde{z})\big)\Big]. \tag{4.24}$$

Grouping the real and imaginary parts we obtain

$$\begin{aligned}\psi(\tilde{z}) = e^{\tilde{z}/2H_s}\Big[&\big(\alpha_R+\beta_R\big)\cos(m\tilde{z}) - \big(\alpha_I-\beta_I\big)\sin(m\tilde{z}) \\ &+ i\Big(\big(\alpha_I+\beta_I\big)\cos(m\tilde{z}) + \big(\alpha_R-\beta_R\big)\sin(m\tilde{z})\Big)\Big].\end{aligned} \tag{4.25}$$

Since we require that ψ be a real function, the imaginary component must vanish. This implies that $\alpha_I = -\beta_I$ and $\alpha_R = \beta_R$ (in other words, A and B are complex conjugates). The result then is

$$\psi(\tilde{z}) = e^{\tilde{z}/2H_s}\Big[C\cos(m\tilde{z}) + D\sin(m\tilde{z})\Big] \tag{4.26}$$

Where $C = 2\alpha_R$ and $D = -2\alpha_I$ are real numbers.

We impose our boundary conditions (3.8) on (4.26). Differentiating (4.26) we obtain

$$\psi'(\tilde{z}) = \frac{1}{2H_s}e^{\tilde{z}/2H_s}\Big[C\cos(m\tilde{z})+D\sin(m\tilde{z})\Big]+e^{\tilde{z}/2H_s}\Big[-mC\sin(m\tilde{z})+mD\cos(m\tilde{z})\Big] \tag{4.27}$$

Writing $k = 1/2H_s$ and rearranging terms gives

$$\psi'(\tilde{z}) = e^{\tilde{z}/2H_s}\left[(kC+mD)\cos(m\tilde{z}) + (kD-mC)\sin(m\tilde{z})\right] \tag{4.28}$$

Together with the boundary conditions $\psi'(\tilde{z}_B) = \psi'(\tilde{z}_T) = 0$, (4.28) gives a system of equations for C and D. We can write this system as

$$\begin{bmatrix} k\cos(m\tilde{z}_B) - m\sin(m\tilde{z}_B) & m\cos(m\tilde{z}_B) + k\sin(m\tilde{z}_B) \\ k\cos(m\tilde{z}_T) - m\sin(m\tilde{z}_T) & m\cos(m\tilde{z}_T) + k\sin(m\tilde{z}_T) \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.29}$$

For a non-trivial solution to exist, the matrix must be singular (i.e., have determinant of zero). The determinant of the coefficient matrix is (after a lot of cancellations)

$$(k^2+m^2)[\sin(m\tilde{z}_T)\cos(m\tilde{z}_B) - \cos(m\tilde{z}_T)\sin(m\tilde{z}_B)] \tag{4.30}$$

$$= (m^2+k^2)\sin(m(\tilde{z}_T - \tilde{z}_B)) \tag{4.31}$$

For (4.31) to vanish, we must have

$$\sin(m(\tilde{z}_T - \tilde{z}_B)) = 0 \tag{4.32}$$

This requires that

$$m(\tilde{z}_T - \tilde{z}_B) = n\pi \tag{4.33}$$

for integer n. Thus, our non-constant solutions of the form (4.26) that also satisfy our boundary conditions – that is to say, our *baroclinic modes* – can be expressed as:

$$\boxed{\psi_n(\tilde{z}) = e^{\tilde{z}/2H_s}\Big[C_n \cos(m_n \tilde{z}) + D_n \sin(m_n \tilde{z})\Big],} \tag{4.34}$$

Where

$$\boxed{m_n = \frac{n\pi}{\tilde{z}_T - \tilde{z}_B},} \tag{4.35}$$

is the vertical wave number, and the coefficients C_n and D_n are the corresponding solutions to (4.29) for the given m_n, indexed by the number of nodes in the vertical profile $n = 1, 2, 3, \ldots$.

Combining (4.35) with (4.22) thus gives us a discrete spectrum of equivalent depths $h_{e,n} = (g\lambda_n)^{-1}$:

$$\boxed{h_{e,n} = \frac{N^2}{g}\left[\left(\frac{n\pi}{\tilde{z}_T - \tilde{z}_B}\right)^2 + \frac{1}{4H_s^2}\right]^{-1}} \tag{4.36}$$

From (4.36), we can observe that the equivalent depth (and thus propagation speed) of the vertical modes increases with N^2; thus, the modes travel faster in more stably stratified conditions. We also observe that the equivalent depth decreases with increasing n; thus, waves that have a smaller wave number (i.e., are longer in the $\tilde{z}$ direction) propagate faster.

4.2.2 Computing the equivalent depths and phase speeds of vertical normal modes

Table 1: Representative parameter values used.

Parameter	Meaning	Representative value
H_s	Reference scale height	7 km
N^2	Buoyancy frequency squared	1×10^{-4} s^{-2}
$\tilde{z}_T$	Domain top (log-pressure height)	18 km
$\tilde{z}_B$	Domain bottom (log-pressure height)	0 km

We can now plot the vertical modes using (4.34) to obtain some insight into the qualitative nature of these modes, as well as to compute the associated equivalent depths and phase speeds of each mode. In all figures and computations that follow in this section, we use the values listed in Table 1

The equivalent depths and phase speeds for the first five baroclinic modes (and the barotropic mode) are listed in Table 2. We can see that higher modes have smaller $h_{e,n}$ and c_n values (Fig. 1). We particularly highlight the fact that the first mode has a phase speed of ∼50 m/s, and the second mode ∼25 m/s. We will give special attention to these first two baroclinic modes in a later section.

Table 2: Equivalent depths and phase speeds for vertical normal modes

n	m_n [m^{-1}]	λ_n [$\mathrm{s}^2\,\mathrm{m}^{-2}$]	$h_{e,n}$ [m]	c_n [$\mathrm{m\,s}^{-1}$]
0	0	0	∞	∞
1	1.75×10^{-4}	3.56×10^{-4}	286.6	53.03
2	3.49×10^{-4}	1.27×10^{-3}	80.3	28.07
3	5.24×10^{-4}	2.79×10^{-3}	36.5	18.92
4	6.98×10^{-4}	4.93×10^{-3}	20.7	14.25
5	8.73×10^{-4}	7.67×10^{-3}	13.3	11.42

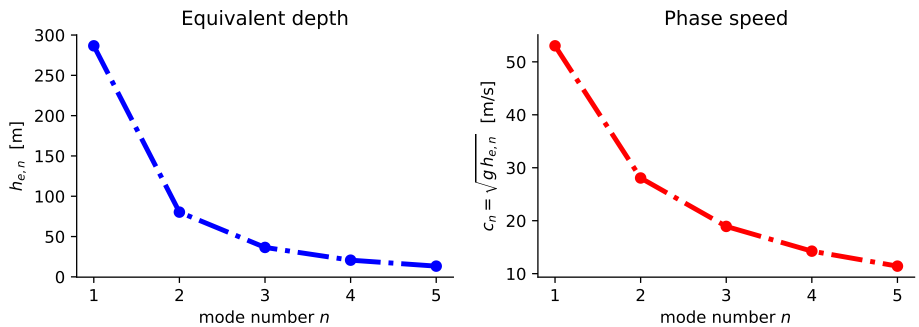


Figure 1: Equivalent depths and phase speeds of idealized vertical normal modes computed from parameters given in Table 1

4.2.3 Plotting the vertical normal modes

We can also compute and plot the actual vertical structure of the normal modes using (4.34). The first few baroclinic modes are plotted in Fig. 2.

Recalling that the modes shown in Fig. 2 correspond to the vertical structures of u, v, and Φ, we can use the hydrostatic equation (2.1d) and the thermodynamic equation (2.1e) to show that the corresponding vertical structures of vertical velocity w and temperature perturbations T are given by the derivatives of the modes in Fig. 2. The derivatives of the first two baroclinic modes are shown in Fig. 3. We see that the first baroclinic mode is associated with a single signed w profile (ascent or descent throughout the troposphere). The second baroclinic mode, by contrast, has a dipole structure, with a crossing point in the middle of the troposphere.

4.3 The baroclinic modes for arbitrary N^2

In the case that $N^2(\tilde{z})$ is not a constant, and instead varies with height, then we cannot use (4.11), and must instead try to solve the full eigenvalue problem (4.1). This cannot be done in closed form for arbitrary $N^2(\tilde{z})$ profiles. Furthermore, in actual application, N^2 is typically reported only at a finite set of heights, and not

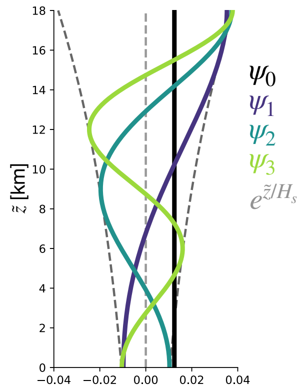


Figure 2: Vertical structures of the barotropic mode ψ_0, and first three baroclinic vertical normal modes $\psi_{1,2,3}$, computed with values given in Table 1. The exponential envelope present in (4.34) is also shown in dashed lines.

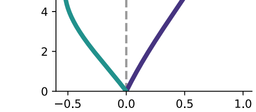


Figure 3: Vertical derivatives of the first two baroclinic vertical normal modes

as a function. Therefore, to obtain quantitative estimates of the vertical normal modes and associated equivalent depths from actual data, we must convert the continuous eigenproblem (4.1) to a discrete eigenproblem. This will involve replacing linear operators with matrices, and functions with vectors.

Let's begin again with the vertical structure operator in log-pressure coordinates

$$L[\psi] = \frac{1}{\rho_0} \frac{\partial}{\partial \tilde{z}} \left(\frac{\rho_0}{N^2} \frac{\partial \psi}{\partial \tilde{z}} \right) \tag{4.37}$$

with boundary conditions (see Exercise 3.5)

$$\left. \frac{\partial \psi}{\partial \tilde{z}} \right|_{\tilde{z}_B} = \left. \frac{\partial \psi}{\partial \tilde{z}} \right|_{\tilde{z}_T} = 0 \tag{4.38}$$

To make the following discussion a little easier to read, and since we anticipate working with data that is typically reported in pressure coordinates (such as in soundings or gridded reanalysis products like ERA5), we can re-write the same operator in isobaric coordinates as

$$L[\Psi] = \frac{\partial}{\partial p} \left(\frac{p}{SR_d} \frac{\partial \Psi}{\partial p} \right) \tag{4.39}$$

with boundary condition

$$\left. \frac{\partial \Psi}{\partial p} \right|_{p_B} = \left. \frac{\partial \Psi}{\partial p} \right|_{p_T} = 0 \tag{4.40}$$

Where $\Psi(p)$ is a function of the vertical pressure coordinate p, and $S = -\frac{T}{\theta}\frac{\partial \theta}{\partial p}$ is the *static stability*.

Exercise 4.3 Show that $L[\Psi]$ as given in (2.9) in log-p height coordinates $\tilde{z}$ is equivalent to Equation 4.39 when converted to pressure coordinates. *Hint:* Recall that $\partial p/\partial \tilde{z} = -p/H_s$, where H_s is a constant scale height

• **Exercise 4.4** Show that product (3.11) in log-p height coordinates $\tilde{z}$ is equivalent, up to a constant factor, to the product

$$\langle f(p), g(p) \rangle = \int_{p_T}^{p_B} f(p)g(p)dp \tag{4.41}$$

Defining $a(p) = p/SR_d$ for convenience, our eigenvalue problem becomes

$$\frac{\partial}{\partial p} \left(a \frac{\partial \Psi}{\partial p} \right) = -\lambda \Psi \tag{4.42}$$

4.3.1 Discretizing the vertical structure operator: the stiffness and mass matrices

To discretize this problem, it is not enough to simply replace the derivatives with finite differences; the reason for this that the orthogonality of our vertical normal modes (expressed in (3.12)) is with respect to a particular choice of product function: the weighted vertical integral. Therefore, we need to discretize both the vertical structure operator L *and* the product (3.11). In order to translate both the operator and product together, we multiply both sides of Equation (4.42) by some "test function" $q(p)$, chosen arbitrarily, and integrate over $[p_T, p_B]$:

$$\int_{p_T}^{p_B} q \frac{\partial}{\partial p}\left(a \frac{\partial \Psi}{\partial p}\right) = -\lambda \int_{p_T}^{p_B} q\Psi \, dp \tag{4.43}$$

Integrating by parts on the left hand side and utilizing boundary conditions (4.40) gives us (Exercise 4.5)

$$\int_{p_T}^{p_B} a(p) \frac{\partial q}{\partial p} \frac{\partial \Psi}{\partial p} \, dp = \lambda \int_{p_T}^{p_B} q\Psi \, dp \tag{4.44}$$

This is the so-called *weak form* of the eigenvalue problem, so called because we require two integrals to be equal, rather than two functions. Note that, to be equivalent to our initial eigenproblem (4.42), we must have (4.44) hold for any given test function q. That is, while the eigenfunction Ψ and the eigenvalue λ are fixed, the test function q is allowed to vary.

The right hand side of (4.44) involves the product of Ψ with our test function q, and the left hand side can be regarded as a functional $k(\phi, q)$ that takes in two functions Ψ and q:

$$k(\Psi, q) = \int_{p_T}^{p_B} a(p) \frac{\partial q}{\partial p} \frac{\partial \Psi}{\partial p} \, dp, \tag{4.45}$$

So (4.44) can be written as

$$k(\Psi, q) = \lambda \langle q, \Psi \rangle \tag{4.46}$$

- **Exercise 4.5** Derive (4.44) from (4.43). *Hint: Use integration by parts, and boundary conditions* (4.40)

We now want to convert (4.46) into a discrete problem that can be handled by a computer. In its current form, (4.46) involves functions with unknown or arbitrary functional form, which cannot be specified in a finite way. However, if we were to specify some finite set of functions, and consider only those functions that can be expressed as linear combinations of these functions, then we would be able to specify any such function with a finite set of coefficients. To be precise, suppose we restrict our space of candidate eigenfunctions $\Psi(p)$ to only those that can be written as

$$\Psi(p) = \sum_{j=0}^{N-1} \hat{\Psi}_j \eta_j(p), \tag{4.47}$$

where $\{\eta_j\}_{j=0}^{N-1}$ are a finite set of basis functions, and $\hat{\Psi}_j$ are expansion coefficients in this basis. Importantly, functions expressible in the form (4.47) can be specified by their expansion coefficients.

For now, we let this basis be arbitrary. In the next subsection we will select a convenient basis for computing our discretized operators explicitly.

Recall that our goal here is to find a function of the form (4.47) that satisfies (4.44) $q(p)$. This condition is equivalent to requiring that (4.44) hold when q is taken to be one of our basis functions η_i (see Exercise 4.6). Therefore, for any test function η_i, we require that

$$\sum_{j=0}^{N-1} \hat{\Psi}_j \underbrace{\int_{p_T}^{p_B} a(p)\, \eta_i'(p)\, \eta_j'(p)\, dp}_{K_{ij}} = \lambda \sum_{j=0}^{N-1} \hat{\Psi}_j \underbrace{\int_{p_T}^{p_B} \eta_i(p)\, \eta_j(p)\, dp}_{M_{ij}} \tag{4.48}$$

- **Exercise 4.6** In (4.48) we only required the weak form (4.44) to hold when the test function q is one of the basis functions η_i. Here we justify that this is equivalent to requiring the weak form to hold for *any* test function we can build from our basis, i.e., any $q = \sum_{i=0}^{n-1} c_i \eta_i$.

  (a) Define the residual

  $$R(q) = \int_{p_T}^{p_B} a \, \frac{\partial q}{\partial p} \frac{\partial \Psi}{\partial p} \, dp - \lambda \int_{p_T}^{p_B} q \, \Psi \, dp, \tag{4.49}$$

  so that the weak form holds for a given q exactly when $R(q) = 0$. Show that R is linear in q: for any functions q_1, q_2 and constants c_1, c_2,

  $$R(c_1 q_1 + c_2 q_2) = c_1 R(q_1) + c_2 R(q_2). \tag{4.50}$$

  Hint: Both integrals are linear in q, since ∂_p and $\int dp$ are each linear.

  (b) Using part (a), show that if $R(\eta_i) = 0$ for every basis function $i = 0, \ldots, n-1$, then $R(q) = 0$ for any $q = \sum_i c_i \eta_i$.

  (c) Argue that the converse is immediate: if the weak form holds for every such q, then in particular it holds for each η_i.

  Hint: What choice of the c_i gives $q = \eta_i$?

Since we require (4.48) to hold for any one of our N basis functions $\{\eta_j\}_{j=0}^{N-1}$, (4.48) gives us a system of N equations for each choice of $i = 0, ..., N-1$. We can express this system as the matrix equation

$$K\vec{\Psi} = \lambda M\vec{\Psi}, \tag{4.51}$$

where

$$\vec{\Psi} = \begin{pmatrix} \hat{\Psi}_0 \\ \hat{\Psi}_1 \\ \vdots \\ \hat{\Psi}_{N-1} \end{pmatrix} \tag{4.52}$$

is the vector composed of the expansion components of Ψ in our basis, and where each entry at row, column (i,j) of the matrices M and K are given by the indicated integral expressions in (4.48).

Equation (4.51) is known as the *generalized eigenvalue problem.* When M is taken to be the identity matrix, then we recover the more standard eigenvalue problem encountered in linear algebra courses. The matrix K is often called the *stiffness matrix*, and M the *mass matrix.* These names come from the fact that an equation of the form (4.51) arises in the context of a system of spring coupled to a set of masses; in this case, K encodes the strength of the interaction between pairs of masses (the stiffness of the system) and M encodes the masses present in the system.

• **Exercise 4.7** Show that M and K are symmetric matrices. That is, for $i \neq j$, show that

$$M_{i,j} = M_{j,i}$$

and

$$K_{i,j} = K_{j,i}$$

4.3.2 Hat functions as a basis set η_i

In the previous section we did not specify a choice of basis functions $\{\eta_i\}$. In this section, we will describe a particularly convenient family of basis functions, the *hat functions*, which will make use of a discretized pressure coordinate grid. That is, we segment our interval $[p_T, p_B]$ into $N-1$ intervals

$$p_T = p_0 < p_1 < \cdots < p_{N-1} = p_B, \tag{4.53}$$

We do not require that the segments be evenly spaced. We label spacing between adjacent levels as $\Delta p_j := p_{j+1} - p_j$.

We now define the *hat functions* $\eta_i(p)$ as the unique continuous, piecewise-linear function on $[p_T, p_B]$ satisfying

$$\eta_i(p_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.54}$$

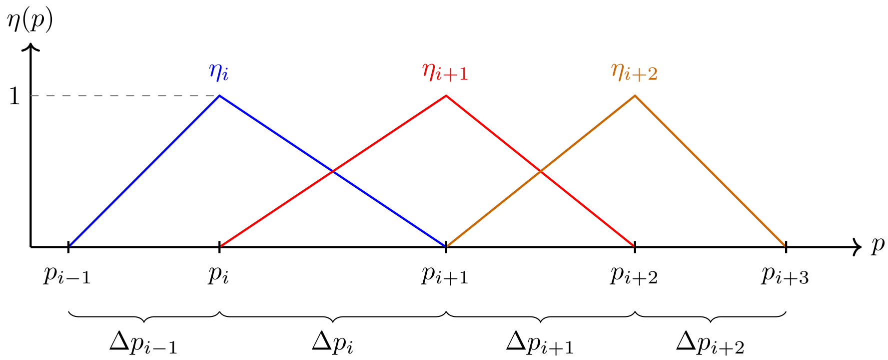


Figure 4: Three adjacent hat functions on a non-uniform pressure grid. Each η_i peaks at value 1 at its associated grid point p_i and vanishes outside the two adjacent intervals. The intervals $\Delta p_j = p_{j+1} - p_j$ are not, in general, equal.

A schematic of these hat functions is shown in Fig. 4. By construction, η_i has value 1 at the grid point p_i, value 0 at every other grid point, and is linearly interpolated between them. From this definition, the derivative of each η is piecewise constant:

$$\eta_i'(p) = \begin{cases} +1/\Delta p_{i-1}, & p \in [p_{i-1}, p_i], \\ -1/\Delta p_i, & p \in [p_i, p_{i+1}], \\ 0, & \text{otherwise} \end{cases} \tag{4.55}$$

We can substitute (4.54) and (4.55) into (4.48) to construct the stiffness matrix K and the mass matrix M in (4.51)

Computing K

For the diagonal elements of K, substituting(4.55) for $i = j$ gives us

$$K_{ii} = \int_{p_{i-1}}^{p_i} \frac{a}{\Delta p_{i-1}^2} dp + \int_{p_i}^{p_{i+1}} \frac{a}{\Delta p_i^2} dp \tag{4.56}$$

If we approximate $a(p)$ over each interval $[p_k, p_{k+1}]$ by its midpoint value, which we denote

$$a_{k+1/2} = \frac{a_k + a_{k+1}}{2}, \tag{4.57}$$

the we can approximate the diagonal elements of K as

$$K_{ii} \approx \frac{a_{i-1/2}}{\Delta p_{i-1}} + \frac{a_{i+1/2}}{\Delta p_i} \tag{4.58}$$

In the case of the off-diagonal elements of K, we make two observations to simplify our calculations. First, we observe that only entries that are one-off the diagonal ($j = i \pm 1$) are non-zero; this is because the hat functions (and by extension their derivatives) only overlap with immediate neighbors (Fig. 4). Also, from the fact that K is a symmetric matrix (Exercise 4.7), it is sufficient to compute $K_{i,i-1} = K_{i-1,i}$

$$K_{i,i-1} = \int_{p_{i-1}}^{p_i} a(p) \times \frac{1}{\Delta p_{i-1}} \times \left(\frac{-1}{\Delta p_{i-1}}\right) dp \approx -\frac{a_{i-1/2}}{\Delta p_{i-1}} \tag{4.59}$$

If we relabel $i \to i+1$, we can obtain an expression for $K_{i+1,i} = K_{i,i+1}$, where we have used the symmetry of K to swap the indices. Thus,

$$K_{i,i+1} = \int_{p_i}^{p_{i+1}} a(p) \times \frac{1}{\Delta p_i} \times \left(\frac{-1}{\Delta p_i}\right) dp \approx -\frac{a_{i+1/2}}{\Delta p_i} \tag{4.60}$$

Since all other elements of K are zero, (4.56), (4.59), and (4.60) completely specify all non-zero entries of K.

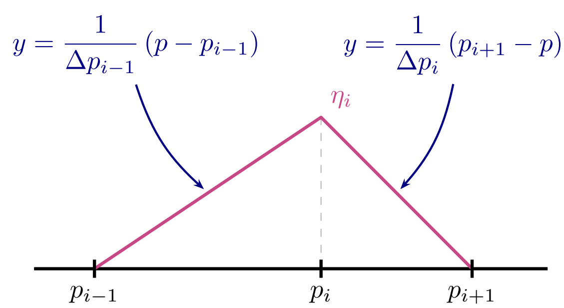


Figure 5: Zoom-in of a single hat function. Linear formulas for each side of the function are indicated.

Computing M

Since M is also symmetric (Exercise 4.7), we can use similar strategy to specify the elements of M. Since the matrix elements of M involve the hat functions themselves, rather than their derivatives (which are piecewise constant), it is useful to explicitly parameterize each side of the hat function with a linear function in p (Fig. 5). Thus, for the diagonal entries of M,

$$M_{ii} = \int_{p_T}^{p_B} \eta_i^2 dp \tag{4.61}$$

$$= \int_{p_i - 1}^{p_i} \frac{1}{\Delta p_{i-1}^2} (p - p_{i-1})^2 \; dp + \int_{p}^{p_{i+1}} \frac{1}{\Delta p_{i+1}^2} (p_{i+1} - p)^2 \, dp \tag{4.62}$$

This integral can be explicitly evaluated to give

$$M_{ii} = \frac{\Delta p_{i-1} + \Delta p_i}{3} \tag{4.63}$$

For the off-diagonal entries, we see that, again, only the one-off diagonal entries are non-zero. Thus, we consider $M_{i,i-1}$:

$$M_{i,i-1} = \int_{p_T}^{p_B} \eta_i \eta_{i-1} \, dp \tag{4.64}$$

$$= \underbrace{\int_{p_{i-2}}^{p_{i-1}} \eta_i \, \eta_{i-1} \, dp}_{\substack{=0 \\ \text{killed by } \eta_i}} + \int_{p_{i-1}}^{p_i} \eta_i \, \eta_{i-1} \, dp + \underbrace{\int_{p_i}^{p_{i+1}} \eta_i \, \eta_{i-1} \, dp}_{\substack{=0 \\ \text{killed by } \eta_{i-1}}} \tag{4.65}$$

The integral simplifies to

$$M_{i,i-1} = \int_{p_{i-1}}^{p_i} \eta_i \, \eta_{i-1} \, dp = \int_{p_{i-1}}^{p_i} \frac{1}{\Delta p_{i-1}} (p - p_{i-1}) \times \frac{1}{\Delta p_{i-1}} (p_i - p) \, dp \tag{4.66}$$

This integral can again be computed explicitly as

$$M_{i,i-1} = \frac{\Delta p_{i-1}}{6} \tag{4.67}$$

Using again a relabelling $i \to i+1$ and the symmetry of M gives us

$$M_{i,i+1} = \frac{\Delta p_i}{6} \tag{4.68}$$

We can thus collect all of the equations for the non-zero elements of K and M:

$$K_{ii} \approx \frac{a_{i-1/2}}{\Delta p_{i-1}} + \frac{a_{i+1/2}}{\Delta p_i}, \tag{4.69a}$$

$$K_{i,i-1} \approx -\frac{a_{i-1/2}}{\Delta p_{i-1}}, \tag{4.69b}$$

$$K_{i,i+1} \approx -\frac{a_{i+1/2}}{\Delta p_i}, \tag{4.69c}$$

$$M_{ii} = \frac{\Delta p_{i-1} + \Delta p_i}{3}, \tag{4.69d}$$

$$M_{i,i-1} = \frac{\Delta p_{i-1}}{6}, \tag{4.69e}$$

$$M_{i,i+1} = \frac{\Delta p_i}{6}, \tag{4.69f}$$

$$a_{k+1/2} = \tfrac{1}{2}\big(a_k + a_{k+1}\big). \tag{4.69g}$$